Unit 5: Graphs of Functions – Revisited

Solving Equations Graphically

The Intersection Method

To solve an equation of the form \( f(x) = g(x) \):

1. Graph \( y_1 = f(x) \) and \( y_2 = g(x) \) on the same screen.
2. Find the \( x \)-coordinate of each point of intersection.

Ex. 1: Solve.

a) \( \frac{2x+1}{x-3} = 4 \)

b) \( x^2 = 3x + 4 \)

c) \( |x^2 - 4x - 3| = x^3 + x - 6 \)

The \( x \)-intercept Method

To solve an equation of the form \( f(x) = g(x) \):

1. Write the equation in the equivalent form \( f(x) = 0 \).
2. Graph \( y = f(x) \).
3. The \( x \)-intercepts of the graph are the real solutions to the equation.

Ex. 2: Solve.

a) \( \frac{2x+1}{x-3} = 4 \)

b) \( x^2 = 3x + 4 \)

c) \( |x^2 - 4x - 3| = x^3 + x - 6 \)

d) \( x^5 + x^2 = x^3 + 5 \)
Technological Quirks

1. Solve $\sqrt{f(x)} = 0$ by solving $f(x) = 0$.
2. Solve $\frac{f(x)}{g(x)} = 0$ by solving $f(x) = 0$ (eliminate any values that also make $g(x) = 0$).

Ex. 3: Solve.

a) $\sqrt{x^4 + x^2 - 2x - 1} = 0$

b) $\frac{2x^2 + x - 1}{9x^2 - 9x + 2} = 0$

Applications

Ex. 1: According to data from the U.S. Bureau of the Census, the approximate population $y$ (in millions) of Chicago and Los Angeles between 1950 and 2000 are given by:

Chicago: $y = .0000304x^3 -.0023x^2 + .02024x + 3.62$

Los Angeles: $y = .0000113x^3 -.000992x^2 + .0538x + 1.97$

where $x = 0$ corresponds to 1950. In what year did the two cities have the same population?

Ex. 2: The average of two real numbers is 41.125, and their product is 1683. Find the two numbers.

Ex. 3: A rectangle is twice as wide as it is high. If it has an area of 24.5 square inches, what are the dimensions of the rectangle?
Ex. 4: A rectangular box with a square base and no top is to have a volume of 30,000 \( cm^3 \). If the surface area of the box is 6000 \( cm^2 \), what are the dimensions of the box?

Ex. 5: A box with no top that has a volume of 1000 cubic inches is to be constructed from a 22 x 30-inch sheet of cardboard by cutting squares of equal size from each corner and folding up the sides. What size square should be cut from each corner?

Ex. 6: A pilot wants to make 840-mile trip from Cleveland to Peoria and back in 5 hours flying time. There will be a headwind of 30 mph going to Peoria, and it is estimated that there will be a 40-tail wind on the return trip. At what constant engine speed should the plane be flown?
Solving Inequalities Graphically

1. Rewrite the inequality in the form $f(x) < 0$ or $f(x) > 0$.
2. Determine the zeros of $f$.
3. Determine the interval(s) where the graph is above ($f(x) > 0$) or below ($f(x) < 0$) the $x$-axis.

Ex. 1: Solve each inequality graphically. Express your answer in interval notation.

a) $x(x + 4)(x - 3)^2 \leq 0$

b) $x^2 - 3x > 4$

c) $\frac{x - 3}{x^2 - 4} < 0$

d) $\frac{3}{x + 4} \leq \frac{2}{x - 1}$

e) $|x^2 + 3x - 4| < 6$

f) $x^4 - 6x^3 + 2x^2 < 5x - 2$

Ex. 2: A company store has determined the cost of ordering and storing $x$ laser printers is:

$$c = 2x + \frac{300,000}{x}$$

If the delivery truck can bring at most 450 printers per order, how many printers should be ordered at a time to keep the cost below $1600.00$?
Increasing, Decreasing and Constant Functions

- A function \( f \) is **increasing** on an interval when, for any \( x_1 \) and \( x_2 \) in the interval, \( x_1 < x_2 \) implies \( f(x_1) < f(x_2) \).
- A function \( f \) is **decreasing** on an interval when, for any \( x_1 \) and \( x_2 \) in the interval, \( x_1 < x_2 \) implies \( f(x_1) > f(x_2) \).
- A function \( f \) is **constant** on an interval when, for any \( x_1 \) and \( x_2 \) in the interval, \( f(x_1) = f(x_2) \).

Ex.1: Determine the open intervals on which each function is increasing, decreasing or constant.

a) \( f(x) = |x - 1| + |x - 3| \)  
b) \( f(x) = x^3 - 3x \)  
c) \( f(x) = x^3 \)
Relative Minimum and Maximum Values (Relative Extrema)

- A function value \( f(a) \) is called **relative minimum** of \( f \) when there exists an interval \((x_1, x_2)\) that contains \( a \) such that \( x_1 < x < x_2 \) implies \( f(a) \leq f(x) \).

- A function value \( f(a) \) is called **relative maximum** of \( f \) when there exists an interval \((x_1, x_2)\) that contains \( a \) such that \( x_1 < x < x_2 \) implies \( f(a) \geq f(x) \).

Ex. 2: Determine the relative minimum and \( x \)-intercepts of \( f(x) = 3x^2 - 4x - 2 \)

Ex. 3: Use a graphing utility to determine the relative minimum and \( x \)-intercepts of \( f(x) = 3x^2 - 4x - 2 \)

Ex. 4: Use a graphing utility to determine any relative minima or maxima for \( f(x) = -x^3 + x \)
Ex. 5: During a 24-hour period, the temperature $t(x)$ (in degrees Fahrenheit) of a certain city can be approximated by the model $t(x) = 0.026x^3 - 1.03x^2 + 10.2x + 34$, $0 \leq x \leq 24$

where $x$ represents the time of day, with $x = 0$ corresponding to 6 A.M.

Approximate the maximum and minimum temperatures during this 24-hour period.

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**Optimization: Translating Words into Functions – revisited**

**Ex. 1:** The sum of two nonnegative numbers is 15. Express the product of one and the square of the other as a function of one of the numbers. Use a graphing utility to find the maximum product.

**Ex. 2:** A rectangle has an area of 400 $in^2$. Express the perimeter of the rectangle as a function of the length of one of its sides. Use a graphing utility to find the minimum perimeter.

**Ex. 3:** An open box is made from a rectangular piece of cardboard that measures 30cm by 40cm by cutting a square of length $x$ from each corner and bending up the sides. Express the volume of the box as a function of $x$. Use a graphing utility to find the dimensions of the box with the maximum volume.
Ex. 4: Express the area of the rectangle as a function of $x$. The equation of the line is $x + 2y = 4$. The lower left-hand corner is on the origin and upper right-hand corner of the rectangle with coordinate $(x, y)$ is on the line. Use a graphing utility to find the rectangle with the maximum area.

![Graph of the line $x + 2y = 4$ with points $(0,0)$ and $(x, y)$]

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**Concavity and Inflection Points**

Concavity is used to describe the way a curve bends. For any two points in a given interval that lie on a curve, if the line segment that connects them is above the curve, then the curve is said to be **concave up** over the given interval. If the segment is below the curve, then the curve is said to be **concave down** over the interval. A point where the curve changes concavity is called an **inflection point**.

![Diagram illustrating concavity and inflection points]
Ex. 1 For the following functions, estimate the following:

1. All local maxima and minima (relative extrema) of the function
2. Intervals where the function is increasing and/or decreasing
3. All inflection points of the function
4. Intervals where the function is concave up and when it is concave down

a) \( f(x) = -2x^3 + 6x^2 - x + 3 \)  
b) \( g(x) = -x^3 + 4x - 2 \)

c) \( f(x) = \frac{3}{(x-2)^2} \)  
d) \( f(x) = \frac{x^2 - x}{x+1} \)