Chapter 9 - Transformations

Transformation: operation that maps (or moves) a preimage onto an image.

4 Types of transformations:
I. Reflections II. Translations III. Rotations IV. Dilations

3 Parts of a transformation:
(1) Pre – image (starting point)
(2) Rule or operation [reflection(flip), translation(slide), or rotation(turn)]
(3) Image (ending point)

TRANSFORMATIONS

\[ \begin{align*}
\text{REFLECTIONS} & \quad \text{TRANSLATIONS} \\
\text{ROTATIONS} & \quad \text{GLIDE REFLECTIONS}
\end{align*} \]

ISOMETRY \[ \rightarrow \] \( \cong \) MAPPING : MAPS EVERY SEGMENT TO A \( \cong \) SEGMENT

PRESERVE:
- DISTANCE (LENGTHS)
- ANGLE MEASURE
- AREA

SIMILARITY MAPPING \[ \rightarrow \] \( \sim \) MAPPING: MAPS ANY POLYGON TO A \( \sim \) POLYGON

~ MAPPINGS PRESERVE:
- ANGLE MEASURE
- ORIENTATION (clockwise/counterclockwise)
- PARALLELISM

FUNCTION NOTATION: One to One Mapping: Function

Example 1: \( g(x) = 3x - 5 \)

a. Find \( g(4) = \) _________

b. Find the image of -5.

c. Find the pre-image of 19.

Example 2: \( h(x, y) \rightarrow (3x, y - 2) \)

a) \( h(3,0) \rightarrow \) _________

b) __________ \( \rightarrow \) (4,-8)
Reflection

A reflection in line \( m \) maps every point \( P \) to a point \( P'^{\perp} \) such that:

- If \( P \) is not on line \( m \), \( m \) is the perpendicular bisector of \( PP'^{\perp} \).
- If \( P \) is on line \( m \), then \( P = P'^{\perp} \).

Notation: \( r_m \) means reflection in line \( m \)

Graph the points to form a figure. Reflect each figure over the \( x \)-axis. Draw the image in a different color. Then write the coordinates of the image points. What pattern do you notice?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>4</td>
<td>-2</td>
<td>-3</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>-6</td>
<td>2</td>
<td>4</td>
<td>-2</td>
</tr>
</tbody>
</table>

Write the algebraic rule for a reflection over the \( x \)-axis:______________________________

Graph the points to form a figure. Reflect each figure over the \( y \)-axis. Draw the image in a different color. Then write the coordinates of the image points. What pattern do you notice?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>-3</td>
<td>2</td>
<td>-3</td>
<td>5</td>
</tr>
<tr>
<td>-5</td>
<td>-3</td>
<td>4</td>
<td>-2</td>
</tr>
</tbody>
</table>

Write the algebraic rule for a reflection over the \( y \)-axis:______________________________
Graph the points to form a figure. Reflect each figure over the line $y = x$ (drawn for you on the first graph). Draw the image in a different color. Then write the coordinates of the image points. What pattern do you notice?

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>4</td>
<td>-2</td>
<td>-3</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>-6</td>
<td>2</td>
<td>4</td>
<td>-2</td>
</tr>
</tbody>
</table>

Write the algebraic rule for a reflection over the line $y = x$:

Write the algebraic rule for a reflection over the line $y = -x$:

Graph the points to form a figure. Reflect each figure over the line $y = -x$ (drawn for you on the first graph). Draw the image in a different color. Then write the coordinates of the image points. What pattern do you notice?

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>-3</td>
<td>2</td>
<td>-3</td>
<td>5</td>
</tr>
<tr>
<td>-5</td>
<td>-3</td>
<td>4</td>
<td>-2</td>
</tr>
</tbody>
</table>

Write the algebraic rule for a reflection over the line $y = -x$:
Examples: Graph the image of the figure using the indicated reflection.

1) reflection across \( y = -2 \)

2) reflection across the x-axis

Use the algebraic rule to find the coordinates of the vertices of the image of each figure after the given reflection.

7. \( D(2, -3), E(1, 4), N(9, -12); \) over the y-axis

8. \( T(-4, 19), H(3, 13), A(5, -6), W(-6, -8); \) over the x-axis

9. \( W(4, -7), I(0, 3), R(-1, -1), E(-5, 5); \) across the line \( x = -4 \)

Write a rule to describe each transformation.
Given: \( A(-2,5) \), \( B(-3,4) \), \( C(4,3) \), \( D(0,-2) \), \( E(-3,6) \) find the following:

a. \( r_x: A \rightarrow \) ________

b. \( r_y: B \rightarrow \) ________

c. \( r_{x=5}: C \rightarrow \) ________

d. \( r_{y=3}: D \rightarrow \) ________

e. \( r_{x=y}: E \rightarrow \) ________

Given: \( r_k: A \rightarrow A^1 \), find the standard form of line \( k \).

a. \( A(5,0) \), \( A^1(9,0) \)

b. \( A(4,1) \), \( A^1(4,3) \)

c. \( A(5,1) \), \( A^1(1,5) \)

d. \( A(0,2) \), \( A^1(4,6) \)

e. \( A(-1,2) \), \( A^1(4,5) \)

Given: \( \Delta MAY \) maps to \( \angle DAY \) by reflecting over \( \overline{AY} \).

a) \( \overline{MY} \) maps to ________.

b) \( \overline{AY} \) maps to ________.

c) \( \angle MAY \) maps to ________.

d) \( \angle AYD \) maps to ________.

e) \( \overline{MA} \) \( \cong \) ________.

f) \( \angle D \) \( \cong \) ________.

g) \( \triangle YAD \) \( \cong \) ________.

6. Find the reflection image of each point over the given reflection line.

a) \( r_{x=4} (4,1) = \) ________

f) \( r_{y=4} (8,4) = \) ________

b) \( r_{x=2} (-3,3) = \) ________

g) \( r_{x=-1} (5,3) = \) ________

c) \( r_{x=2} (7,2) = \) ________

h) \( r_{x=3} (8,-2) = \) ________

d) \( r_{y=-1} (4,2) = \) ________

i) \( r_{x=1} (-3,4) = \) ________

e) \( r_{y=-2} (3,2) = \) ________

j) \( r_{y=2} (-3,-2) = \) ________
Rotation

Counterclockwise is POSITIVE (think of the quadrants).
Clockwise is NEGATIVE

Notation for Rotations:
- \( R_{O, 270^\circ} \) means a counterclockwise rotation about point O of 270˚.
- \( R_{O, -90^\circ} \) means a clockwise rotation about point O of 90˚.
- \( H_O \) means a rotation about point O of 180˚. (Half-turn)

\( H_O \) is equivalent to ________ and ________.

Find an equivalent rotation for the following:
1. \( R_{O, 50} \)
2. \( R_{O, -40} \)
3. \( R_{O, -90} \)
4. \( R_{O, -180} \)

In the diagram, O is the center of equilateral \( \Delta PST \).
State the images of points P, S, and T for each rotation.
5. \( R_{O, 120} \)
6. \( R_{O, -120} \)
7. \( R_{O, 360} \)

Name each image point:
8. \( R_{T, 60}(S) = \)
9. \( R_{T, -60}(P) = \)
10. \( R_{O, 240}(P) = \)

The diagonals of regular hexagon ABCDEF form 6 equilateral \( \Delta \)'s. Complete the following:
11. \( R_{O, 60}: E \rightarrow \) ________
12. \( R_{O, -60}: D \rightarrow \) ________
13. \( R_{O, 120}: F \rightarrow \) ________
14. \( R_{D, 60}: \) _______ \( \rightarrow \) O
15. \( R_{B, -60}(O) = \) ________
16. \( H_O (A) = \) ________
17. A reflection in \( \overline{FC} \) maps B to _____ and D to _____.
18. If k is the perpendicular bisector of \( \overline{FE} \), then \( R_k(A) = \) ________.
19. If a translation maps A to B, then it also maps O to _____ and E to _____.

Examples:

1. ABCDE is a regular pentagon.
   Name the image of point E for a counterclockwise 72 degree rotation about X.

2. Name the image of point A for a clockwise 216° rotation about X.
The hexagon GIKMPR and ∆FJN are regular. The dashed line segments form 30° angles.

3. Find the angle of rotation about O that maps Q to F.

4. Find the image of point P after a rotation of 240° about point O.

5. Find the angle of rotation about O that maps JK to FG.

Graph the image of the figure and give its coordinates.

1) rotation 90° clockwise about the origin
U(1, -2), W(0, 2), K(3, 2), G(3, -3)

rotation 180° about the origin
V(2, 0), S(1, 3), G(5, 0)

rotation 180° about the origin
rotation 90° counterclockwise about the origin
rotation 90° clockwise about the origin
Translations

You can write rules to describe a translation.

\[ T(x, y) \rightarrow (x + a, x + b) \] moves a figure “a” units right and “b” units up.

\[ T(x, y) \rightarrow (x - a, x - b) \] moves a figure “a” units left and “b” units down.

Example 1 - On the coordinate plane, graph PQRS where P(-1, -2) Q(1, 2) R(-2, 2) S(1, -2). Use the rule \((x, y) \rightarrow (x + 3, y - 1)\) to find the translation of the image of PQRS.

Example 2 - Write the rule to describe each translation.

You can use a vector to describe a translation.

Example 3 - \(\langle 2, 3 \rangle\) means to go ________________ \(\langle -1, 2 \rangle\) means to go ________________

\(\langle 0, -2 \rangle\) means to go ________________

Example 4 - Find the image of \(\triangle ACE\) with vertices A(9,3) C(-3, -2) E(0, -7) under the translation \(\langle -5,4 \rangle\).

Example 5 - What is the image of F(11, -2) under the translation \(\langle -6, -8 \rangle\)?

Example 6 - The point Q’(7, -1) is the image under the translation \(\langle 5, -2 \rangle\). What is the preimage Q?

Example 7 - B(-6, -9) maps to B’(-12, 3). Describe the rule and vector.

Each triangle is an isosceles right triangle.

1. If A maps to B, then \(G \rightarrow \) ______ and \(E \rightarrow \) ______ and \(\overline{AD} \rightarrow \) ______.

2. If C maps to A, then \(I \rightarrow \) ______ and \(\overline{IF} \rightarrow \) ______.

3. If E maps to H, then \(\overline{AD} \rightarrow \) ______ and \(\overline{FC} \rightarrow \) ______.

4. If I maps to H, then \(\overline{BC} \rightarrow \) ______ and \(\overline{EH} \rightarrow \) ______.
True or False, if false, explain why.

1. A translation is a transformation. \(\text{True}\)

2. Translations preserve angle measure. \(\text{True}\)

3. Translations reverse orientation. \(\text{False}; \text{ they preserve orientation.}\)

4. Under a translation, a line is parallel to its image. \(\text{True}\)

5. Under every line reflection, a line is parallel to its image. \(\text{False}; \text{ lines are perpendicular.}\)

6. If \(P'\) and \(Q'\) are the translation images of \(P\) and \(Q\), then \(PP' = QQ'\). \(\text{True}\)

7. If \(S'\) and \(T'\) are the reflection images of \(S\) and \(T\), then \(SS' = TT'\). \(\text{False}; \text{ these are congruent squares.}\)

8. The image of a square under a translation is a square. \(\text{True}\)

9. A point may be its own image under a translation. \(\text{True}\)

10. A point may be its own image under a line reflection. \(\text{True}; \text{ for example, the point at the intersection of the line of reflection and the plane.}\)

11. In the tessellation to the right, a translation maps \(A\) onto \(B\). Name the image of:

   a. \(E\)  
   b. \(K\)  
   c. \(B\)  
   d. \(\overrightarrow{AE}\)  
   e. \(\angle EFK\)  
   f. \(BCGF\)

12. Using the same diagram, in a translation that maps \(A\) onto \(F\), name the image of:

   a. \(E\)  
   b. \(K\)  
   c. \(B\)  
   d. \(\overrightarrow{AE}\)  
   e. \(\angle EFK\)  
   f. \(BCGF\)

In 14 – 25, refer to the graph. A translation maps \(A\) onto \(A'\). Name the coordinates of the translation image of each of the following points.

14. \(O\)  
15. \(B\)  
16. \(C\)  
17. \(D\)  
18. \(E\)  
19. \(A'\)

Name the coordinates of the pre-image of each of the following points.

20. \(A'\)  
21. \(O\)  
22. \(B\)  
23. \(C\)  
24. \(D\)  
25. \(E\)
Dilations

The scale factor is usually called "k". A dilation is NOT an isometry.

If $0 < k < 1$, then the dilation is a ________________.

If $k > 1$ then the dilation is an ________________.

Mapping Rule: $D_k (x, y) \rightarrow (kx, ky)$

$\begin{align*}
    k &= \frac{\text{length of side of image}}{\text{length of side of pre-image}} \\
    k &= \frac{x\text{-coordinate from image}}{x\text{-coordinate from pre-image}}
\end{align*}$

Example 1 - The dashed triangle is a dilation image of the solid triangle. What is the scale factor?

Example 2 - A dilation has center (0,0). Find the image of each point for the scale factor given.

   a) $D(2, -5): 2$
   b) $A(-6, -2): 1.5$
   c) $T(0, 6): 3$
   d) $C(-4, -7): 0.1$

Example 3 - WXY is dilated by $2/3$. W(2,5), X(6,2), Y (2,2) Find X'.

Example 4 - Find the vertices for the image of $\triangle ABC$ with vertices $A (4,-2), B (-2, 5), C (-3,-1)$. Use the rule $(x,y) \rightarrow (2x, 2y)$. What is the scale factor?

Example 5 - Write the rule to describe the dilation. A (-6, 9), B(3, 12), C(0,-6) What is the scale factor? A'(-2, 3), B'(1, 4), C'(0,-2)

Example 6 - What is the ratio of side lengths between the preimage and image after the dilation $D_4$?

Example 7 - Square ABCD has vertices $A(-4,4), B(4,4), C(4,-4)$, and $D(-4,-4)$. Write the coordinates of the image after a dilation of the figure by a scale factor of 4 about the origin.

Example 8 - Find the image after a dilation of 2 for $\triangle ABC$. A(3,-1), B(-5,4), and C(2,-6).
Example 9: a) Given A(0, 0)  B(2, 2)  C(-1, 2)  D(-3, 0), find the image of the figure with a dilation of 3. Graph the preimage and the image.

b) Find the image of the figure with a dilation of 1/2.

Example 10: What is the rule for a scale factor of 3 and left 2 units?

A. \((x', y') = (3x, 3y)\)  
B. \((x', y') = (3x + 2, 3y)\)

C. \((x', y') = (3x, 3y - 2)\)  
D. \((x', y') = (3x - 2, 3y)\)

Bonus: Given R(4,1), S(7,6), and T(9,3) If \(T(x', y') = (2x - 3, 2y + 4)\) Find R'.